Vector Spaces

(Sections 5.1/6.1 and 6.2)

Explanation: the applications later in the course (eigenvalues, projections/best-fit lines) won't make sense unless we develop appropriate terminology.

This is the terminology!

The idea: you're used to vectors being either rows or columns of real numbers. For vector spaces in general, our focus will be on <u>properties</u>: you can add any two column (or row) vectors and scalar multiply them. For us, a vector space will be anything where we can define <u>"addition"</u> and <u>"scalar multiplication"</u>.

Example 1: ($\mathcal{F}(\mathbb{R})$) Let $\mathcal{F}(\mathbb{R})$ denote all functions f of a real number x that output a real number $y = f(x)$.

We write $f: \mathbb{R} \to \mathbb{R}$.

We want to "add" and "scalar multiply" functions.

To do this, we'll specify what has to happen for every x in $\mathbb{R}$ since functions are determined by their values on inputs.

Let x be a real number and let f and g be functions in $\mathcal{F}(\mathbb{R})$.

If c is a real number, then we define the functions $f+g$ and $c \cdot f$ by

$$\underline{(f+g)(x)} = \overset{\text{real number}}{\overset{\downarrow}{f(x)}} + \overset{\text{real number}}{\overset{\downarrow}{g(x)}}$$

$$\underline{(c \cdot f)(x)} = \underset{\text{real number}}{\underset{\uparrow}{c}} \cdot \underset{\text{real number}}{\underset{\uparrow}{f(x)}}$$

This is an example of an abstract vector space: vectors are now functions $f: \mathbb{R} \to \mathbb{R}$!

Vector Spaces in General

A vector space V is a collection of objects such that you can define two operations

- addition of vectors "$+$"
- scalar multiplication of vectors "$\cdot$"

where if x, y are objects in V, then

$x + y$ is also an object in V and

$c \cdot x$ is also an object in V for all scalars c AND for all objects x, y, z in V, which we will call vectors, and all scalars c and d,

1) $(x+y)+z = x+(y+z)$

associativity of addition

2) $c\cdot(d\cdot x) = (c\cdot d)\cdot x$

associativity of scalar multiplication

3) $x+y = y+x$

commutativity of addition

4) $(c+d)\cdot x = c\cdot x + d\cdot x$

$c\cdot(x+y) = c\cdot x + c\cdot y$

distributivity of scalar multiplication over addition

5) $1\cdot x = x$

6) There is a vector 0_V in V such that

$$0_V + x = x.$$

existence of additive identity

7) For each x, there is a vector $-x$ in V such that

$$x + (-x) = 0_V$$

existence of additive inverses

We never want to check all these properties!

We'd like to assemble a catalog of vector spaces where we know what "+", "·", and 0_V are.

Example 2: ($\mathbb{R}^n$ and $M_{n\times 1}(\mathbb{R})$)

These are all vector spaces:

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

are column vectors in $\mathbb{R}^n$, then

if c is a scalar, we know that

$$x+y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$c\cdot x = \begin{bmatrix} c\cdot x_1 \\ c\cdot x_2 \\ \vdots \\ c\cdot x_n \end{bmatrix}, \quad \text{zero vector} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If $A = (A_{i,k})$, $B = (B_{i,k})$ are $m \times n$ matrices and c is a scalar,

$$(A+B)_{i,k} = A_{i,k} + B_{i,k}$$

$$(c \cdot A)_{i,k} = c \cdot A_{i,k}$$

for all $1 \leq i \leq m$, $1 \leq k \leq n$.

The zero vector is the zero matrix $A_{i,k} = 0$ for all $1 \leq i \leq m$, $1 \leq k \leq n$.

Example 3: (sequences) Let $\mathcal{S}$ denote the collection of all sequences of real numbers. If $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are such sequences and $c \in \mathbb{R}$, we can define addition and scalar multiplication of sequences by

$$(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty} = (a_n + b_n)_{n=1}^{\infty}$$

$$c \cdot (a_n)_{n=1}^{\infty} = (c \cdot a_n)_{n=1}^{\infty}$$

The zero vector is the zero sequence $a_n = 0$ for all $n \geq 1$:

$$(0,0,0,\ldots) = 0_V.$$

With these operations, $\mathcal{S}$ becomes a vector space!

Note: (zero function for $\mathcal{F}(\mathbb{R})$)

The zero vector for $V = \mathcal{F}(\mathbb{R})$ is the function $f(x) = 0$ for all real numbers x (graph is a horizontal line $y = 0$).

Summary

Vectors in a vector are whatever I can define addition and scalar multiplication for:-

If $V = \mathbb{R}^n$, vectors are row or column arrays of real numbers

If $V = M_{m \times n}(\mathbb{R})$, vectors are $m \times n$ arrays of real numbers (matrices)

If $V = \mathcal{F}(\mathbb{R})$, vectors are functions from $\mathbb{R}$ to $\mathbb{R}$

If $V = \mathcal{S}$, vectors are sequences of real numbers

Problem: Checking all those properties!

Solution: Cheat, by looking inside things we already know to be vector spaces!

Some mathematical notation:

"$\in$" = "is a member of"

"$\subseteq$" = "contained in and possibly equal"

"$\forall$" = "for every"

"$\exists$" = "there is"

"$\Rightarrow$" = "implies"

Subspaces

Take a vector space V and let $W \subseteq V$. We say that W is a **subspace** of V if W is also a vector space under the operations of addition and scalar multiplication from V.

Note: we need W to actually have vectors in it.

The Subspace Test

If V is a vector space, $W \subseteq V$ is a subspace precisely when

1) $0_V \in W$

2) $\forall\ x, y \in W$, then $x+y \in W$

3) $\forall\ c \in \mathbb{R}, x \in W$, then $c \cdot x \in W$

Note: "$+$" and "$\cdot$" are the addition and scalar multiplication in V.

Example 4: (a plane in $\mathbb{R}^3$) If $V = \mathbb{R}^3$, let $W \subseteq V$, W will be all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$\underbrace{7x - 11y + 210z = 0}_{\text{defining property of } W}.$$

Show that W is a subspace of $\mathbb{R}^3$

Solution: Use the subspace test.

1) $0_V \in W$. What is 0_V?

In this case, $0_V = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$0_V \in W$ means that 0_V satisfies the defining property of W:

$$7x - 11y + 210z = 0.$$

Plug in $x = y = z = 0$ and check that this gives you zero in the above equation:

$$7 \cdot 0 - 11 \cdot 0 + 210 \cdot 0$$

$$= 0 - 0 + 0 = 0 \checkmark$$

So $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$.

2) If $\vec{x}, \vec{y} \in W$, then

$\vec{x} + \vec{y} \in W$.

This means: if $\vec{x}$ and $\vec{y}$ satisfy the defining property of W, then so does $\vec{x}+\vec{y}$.

If $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$,

then $\vec{x}$ and $\vec{y}$ satisfying the defining property of W means

$$7x_1 - 11y_1 + 210z_1 = 0 \quad \text{and}$$

$$7x_2 - 11y_2 + 210z_2 = 0$$

Then $\vec{x}+\vec{y} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

Check that this satisfies the defining property of W.

$$7(x_1+x_2) - 11(y_1+y_2) + 210(z_1+z_2)$$

group

$$= 7x_1 + 7x_2 - 11y_1 - 11y_2 + 210z_1 + 210z_2$$

group

$$= \underbrace{(7x_1 - 11y_1 + 210z_1)}_{=0} + \underbrace{(7x_2 - 11y_2 + 210z_2)}_{=0}$$

$$= 0 + 0 = 0$$ ✓

Then $\vec{x}+\vec{y}$ satisfies the defining property of W, so $\vec{x}+\vec{y} \in W$.

3) Let $\vec{x} \in W$, $c \in \mathbb{R}$. Show $c \cdot \vec{x} \in W$.

To show $c \cdot \vec{x} \in W$, we need to show $c \cdot \vec{x}$ satisfies the defining property of W.

Since $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in W$, we know $\vec{x}$ satisfies the defining property of W:

$$7x_1 - 11y_1 + 210z_1 = 0$$

But $c \cdot \vec{x} = c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}$

Show that this satisfies the defining property of W

$$7(cx_1) - 11(cy_1) + 210(cz_1)$$

$$= c(\underbrace{7x_1 - 11y_1 + 210z_1}_{=0})$$

$$= c \cdot 0$$

$$= 0 \checkmark$$

This shows $c \cdot \vec{x} \in W$ whenever $\vec{x} \in W$ and $c \in \mathbb{R}$.

This shows W is a subspace of $\mathbb{R}^3$ by the subspace test!

Remark: (span) If V is a vector space and $S \subseteq V$ is a collection of vectors, we can define, as for $\mathbb{R}^n$,

$$\operatorname{span}(S) = \text{all linear combinations of vectors } \underline{\text{in } S}$$

$$= \text{all sums of scalar multiples of vectors } \underline{\text{in } S}.$$

For any $S \subseteq V$ such that S actually contains vectors, <u>span(S) is a subspace of V.</u> You may use this fact whenever appropriate.

Two-Step Subspace Test

If V is a vector space, $W \subseteq V$ is a subspace precisely when

1) $0_V \in W$

2) $\forall\ x, y \in W,\ c \in \mathbb{R},$

$cx + y \in W$.

Again, we need W to actually contain vectors.

Example 5: (upper-triangular matrices)

Let $V = M_{n\times n}(\mathbb{R})$. Let $W \subseteq V$ be the collection of all $A = (A_{i,k})$ such that $\underbrace{A_{i,k} = 0 \text{ if } i > k}_{\text{defining property of } W}$.

These matrices look like

$$n=2 \qquad \begin{array}{c} \begin{array}{cc} 1,1 & 1,2 \end{array} \\ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \\ \begin{array}{cc} 2,1 & \end{array} \end{array}$$

$$n=3 \qquad \begin{array}{c} \begin{bmatrix} a & b & c \\ {}_{2,1}\,0 & e & f \\ 0 & 0 & g \end{bmatrix} \\ \begin{array}{ccc} 3,1 & 3,2 & \end{array} \end{array} \qquad \text{etc.}$$

a, b, c, e, f, g can be any real number.

W is called the collection of upper-triangular matrices.

Show W is a subspace of $M_{n\times n}(\mathbb{R})$.

Use the two-step subspace test.

1) $0_V \in W$.

Here, $0_V = 0_{n\times n}$, the $n\times n$ zero matrix, where every entry is zero!

In particular,

$(0_{n\times n})_{i,k} = 0$ if $i>k$.

This shows $0_{n\times n}$ satisfies the defining property of W, so $0_{n\times n} \in W$.

2) Let $A, B \in W$ and $c \in \mathbb{R}$. Show $cA+B \in W$.

Since $A, B \in W$, they satisfy the defining property of W:

$$A = (A_{i,k}) \text{ and } A_{i,k} = 0 \text{ if } i > k$$

$$B = (B_{i,k}) \text{ and } B_{i,k} = 0 \text{ if } i > k$$

Now

$$(cA+B)_{i,k} = (cA)_{i,k} + B_{i,k}$$

$$= c A_{i,k} + B_{i,k}$$

$$= c \cdot 0 + 0 \quad \boxed{\text{if } i > k}$$

$$= 0$$

We have shown that $cA+B$ satisfies the defining property of W, so

$cA+B \in W$. By the two-step subspace test, W is a subspace of $V = M_{n \times n}(\mathbb{R})$.

Example 6: (not a subspace) Let $V = \mathbb{R}^2$ and let $S \subseteq V$ be all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $xy \geq 0$. Show S is not a subspace of $\mathbb{R}^2$.

Solution: All we need is to come up with either

1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in S

or

2) $\vec{x}, \vec{y} \in S$ but $\vec{x} + \vec{y} \notin S$

for <u>explicit</u> examples of $\vec{x}$ and $\vec{y}$,

or

3) $\vec{x} \in S$ but $c \cdot \vec{x}$ is not in S for an explicit choice of $\vec{x}$ and c.

Try

1) zero vector: $x = y = 0$

$$0 \cdot 0 = 0 \geq 0$$, so

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$$ won't work

2) We need a choice of $\vec{x}$ and a choice of $\vec{y}$ where $\vec{x}, \vec{y} \in S$ but $\vec{x} + \vec{y} \notin S$.

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S$$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$$

not a counterexample

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in S, \quad \begin{bmatrix} -1 \\ -4 \end{bmatrix} \in S$$

But $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$1 \cdot (-1) = -1 < 0$, so

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not in S.

So S is not a subspace of $\mathbb{R}^2$.